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ON THE THEORY OF THE GYROPENDULUM*

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Precession equations of motion of the gyropendulum relative to the accompanying Darboux trihedron /1/ and, also, precession equations of the gyropendulum motion relative to the geographic trihedron, considered in /2,3/, are given a kinematic interpretation. Linear differential equations that define the gyropendulum behavior at finite deflection angles of the rotor axis from the vertical are established for arbitrary motions of its suspension point over the surface of the Earth. These equations have the form of kinematic equations of a solid body spherical motion in terms of Rodrigues-Hamilton parameters, and in the case of stationary base they are in agreement with equations established in /4/. The Liapunov stability ot the gyropendulum equations in both the finite Euler—Krylov angles and in the Rodrigues —Hamilton parameters is proved. Particular cases of integrability in quadratures of the gyropendulum precession equations at finite angles are indicated.

1. Let us consider the gyropendulum motion relative to the natural Darboux trihedron $Ox^\circ y^\circ z^\circ$ with its vertex at the center of the Cardan universal suspension joint with its edge z° normal to the Earth surface and edge x° directed along the suspension center velocity vector $\mathbf{v}$ /1,5/.

The input precession equations of motion of a gyropendulum for arbitrary motion of its suspension point over the Earth surface are of the form

$$H\omega_x' = a(P_x + F_x),\ H\omega_y' = a(P_y + F_y) \tag{1.1}$$

where ω_x', ω_y' are projections of the absolute angular velocity $\boldsymbol{\omega}'$ of the system of coordinates $Oxyz$ with its origin at the Cardan suspension center on its axes and axis z directed along the gyropendulum axis, H is the gyroscope intrinsic moment; P_x, P_y and F_x, F_y are projections of the resultant $\mathbf{P}$ of transfer inertia forces and of gravity force $\mathbf{F}$ (these forces pass through the system "inner ring-rotor" center of gravity) on the axes of the system of coordinates $Oxyz$, and a is the distance between the center of gravity and the coordinate origin O.

We determine the position of the coordinate system $Oxyz$ relative to the trihedron $Ox^\circ y^\circ z^\circ$ by the angles α, β, γ (Fig.1). Angles α and β define the position of the gyropendulum rotor axis z relative to the trihedron $Ox^\circ y^\circ z^\circ$. Below are given the cosines of angles between the axes of the coordinate system $Oxyz$ and the edges of the trihedron $Ox^\circ y^\circ z^\circ$

$$\begin{array}{cccc} & x^\circ & y^\circ & z^\circ \\ x & \cos\beta\cos\gamma & \sin\alpha\sin\beta\cos\gamma+\cos\alpha\sin\gamma & -\cos\alpha\sin\beta\cos\gamma+\sin\alpha\sin\gamma \\ y & -\cos\beta\sin\gamma & -\sin\alpha\sin\beta\sin\gamma+\cos\alpha\cos\gamma & \cos\alpha\sin\beta\sin\gamma+\sin\alpha\cos\gamma \\ z & \sin\beta & -\sin\alpha\cos\beta & \cos\alpha\cos\beta \end{array} \tag{1.2}$$

We denote by $\boldsymbol{\omega}$ the vector of angular velocity of the coordinate system $Oxyz$ rotation relative to the trihedron $Ox^\circ y^\circ z^\circ$. The projections $\omega_i\ (i=1,2,3)$ of this vector on the axes of the coordinate system $Ox^\circ y^\circ z^\circ$ are defined by formulas

$$\omega_1 = \alpha^\cdot + \gamma^\cdot \sin\beta,\ \omega_2 = \beta^\cdot\cos\alpha - \gamma^\cdot\cos\beta\sin\alpha,\quad \omega_3 = \beta^\cdot\sin\alpha + \gamma^\cdot\cos\beta\cos\alpha \tag{1.3}$$

from which we obtain

$$\alpha^\cdot = \omega_1 - \operatorname{tg}\beta\,(\omega_3\cos\alpha - \omega_2\sin\alpha),\ \beta^\cdot = \omega_2\cos\alpha + \omega_3\sin\alpha,\quad \gamma^\cdot = (\omega_3\cos\alpha - \omega_2\sin\alpha)/\cos\beta \tag{1.4}$$

We select the motion of the coordinate system $Oxyz$ with respect to coordinate γ so that the projection ω'_z of vector $\boldsymbol{\omega}'$ on the z-axis be determined by formula

$$\omega'_z = a(P_z + F_z)/H \tag{1.5}$$

where P_z and F_z are projections of vectors $\mathbf{P}$ and $\mathbf{F}$ on the z-axis. The scalar relations (1.1) and (1.5) are then equivalent to the single vector relation

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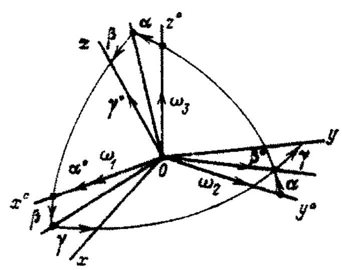

Fig.1

$$\omega' = a(\mathbf{P}+\mathbf{F})\,/\,H = a(\mathbf{F}-m\mathbf{w})/H \tag{1.6}$$

where m is the over-all mass of the inner ring and rotor of the gyropendulum, and $\mathbf{w}$ is the absolute acceleration of the gyropendulum suspension point /1,5/.

The vector of relative angular velocity of rotation of the coordinate system $Oxyz$ is $\omega = \omega' - \omega^\circ$, where ω° is the absolute angular velocity of rotation of the trihedron $Ox^\circ y^\circ z^\circ$. Using formula (1.6) we obtain

$$\omega = a(\mathbf{F}-m\mathbf{w})/H - \omega^\circ \tag{1.7}$$

Projecting the left- and right-hand sides of the vector equality (1.7) on the edges of the trihedron $Ox^\circ y^\circ z^\circ$ and taking into account expressions for the projections of vectors $\mathbf{F}$, $\mathbf{w}$, and ω° on the edges of the trihedron $Ox^\circ y^\circ z^\circ$ /1,5/, for the projections ω_i of vector ω we obtain

$$\omega_1 = -\frac{am}{H}v^{\cdot},\quad \omega_2 = -\frac{am}{H}\omega_0 v - \frac{v}{R},\quad \omega_3 = -\frac{a}{H}\left(F - \frac{mv^2}{R}\right) - \omega_0\quad \left(\omega_0 = \frac{v}{\rho}\right) \tag{1.8}$$

where v is the velocity of the gyropendulum suspension point relative to the nonrotating sphere S with the same center and radius as the Earth, R is the Earth radius, and ρ is the radius of geodesic curvature of the gyropendulum suspension point trajectory, where that point is located at a given instant of time.

Formulas (1.8) and the first two equations of system (1.4) yield the equations of precession motion of the z-axis of the gyropendulum rotor relative to the Darboux trihedron for an arbitrary motion of its suspension point over the Earth surface /1,5/

$$H(vR^{-1}\sin\alpha\sin\beta - \omega_0\cos\alpha\sin\beta + \alpha^{\cdot}\cos\beta) = a[(F - mv^2R^{-1})\cos\alpha\sin\beta - m(v^{\cdot}\cos\beta + \omega_0 v\sin\alpha\sin\beta)] \tag{1.9}$$

$$H(vR^{-1}\cos\alpha + \omega_0\sin\alpha + \beta^{\cdot}) = a[(mv^2R^{-1} - F)\sin\alpha - m\omega_0 v\cos\alpha]$$

Taking into account equality (1.8) we reduce the third equation of system (1.4) to the form

$$\gamma^{\cdot} = \frac{1}{\cos\beta}\left\{\left[\frac{a}{H}\left(\frac{mv^2}{R} - F\right) - \omega_0\right]\cos\alpha + \left(\frac{am}{H}\omega_0 v + \frac{v}{R}\right)\sin\alpha\right\} \tag{1.10}$$

which, after the determination of the unknown functions of time $\alpha = \alpha(t)$ and $\beta = \beta(t)$, enables us to establish the law of variation of coordinate $\gamma = \gamma(t)$ for which equality (1.5) and, consequently also, the vector equality (1.6) are valid.

2. We attach to the system of coordinates $Oxyz$ a solid body D locating one of its points at the origin O of the coordinates system $Oxyz$. Let the body D effect a spherical motion at angular velocity ω defined by formula (1.7) relative to the Darboux trihedron $Ox^\circ y^\circ z^\circ$. The kinematic equations of that motion are of the form (1.4).

We introduce vector θ of final turn which defines the position of body D relative to the trihedron $Ox^\circ y^\circ z^\circ$, and denote the Rodrigues—Hamilton parameters that correspond to the final turn vector θ by $\lambda_j\ (j = 0, 1, 2, 3)$. Parameters λ_j are determined by the angles α, β, γ in conformity with the known formulas /4/. To obtain formulas for angles α, β, γ in terms of the Rodrigues—Hamilton parameters λ_j, we define the expressions (1.2) in terms of parameters λ_j in conformity with the formulas given in /6/ and obtain

$$\sin\beta = 2(\lambda_0\lambda_2 + \lambda_1\lambda_3),\quad \operatorname{tg}\alpha = 2(\lambda_0\lambda_1 - \lambda_2\lambda_3)\,/\,(\lambda_0^2 + \lambda_3^2 - \lambda_1^2 - \lambda_2^2) \tag{2.1}$$
$$\operatorname{tg}\gamma = 2(\lambda_0\lambda_3 - \lambda_1\lambda_2)\,/\,(\lambda_0^2 + \lambda_1^2 - \lambda_2^2 - \lambda_3^2)$$

The kinematic equations of spherical motion of body D that relate the Rodrigues—Hamilton parameters and their derivatives to projections ω_i of vector ω of the body relative angular velocity on the axes of the coordinate system $Ox^\circ y^\circ z^\circ$ are of the form /6,7/

$$2\lambda_0^{\cdot} = -(\omega_1\lambda_1 + \omega_2\lambda_2 + \omega_3\lambda_3),\quad 2\lambda_1^{\cdot} = \omega_1\lambda_0 + \omega_2\lambda_3 - \omega_3\lambda_2 \tag{2.2}$$
$$2\lambda_2^{\cdot} = \omega_2\lambda_0 + \omega_3\lambda_1 - \omega_1\lambda_3,\quad 2\lambda_3^{\cdot} = \omega_3\lambda_0 + \omega_1\lambda_2 - \omega_2\lambda_1$$

It was shown above that the dynamic equations (1.9) of the gyropendulum precession motion are equivalent to the first two kinematic equations (1.4) and formulas (1.8). Equations (1.4) are, in turn, equivalent to Eqs. (2.2), since both represent in different form the kinematic equations of spherical motion of one and the same body D. Hence, when ω_i are specified by formulas (1.8), Eqs. (2.2) can be treated as equations of precession motion of the gyropendulum rotor axis z relative to the Darboux trihedron. These equations, as well as Eqs. (1.9) are valid for any arbitrary motion of the gyropendulum suspension point over the Earth surface, provided the conditions stipulated in /1,5/ imposed on the forces acting on the gyropendulum are satisfied.

For determining angles α and β that define the position of the gyropendulum rotor axis z relative to the Darboux trihedron for arbitrary functions $v(t)$ and $\omega_0(t)$ it is, thus, possible to solve four linear differential equations (2.2) instead of two essentially nonlinear differential equations (1.9), by passing from parameters λ_j to angles α and β in conformity with the first two formulas (2.1). The third formula (2.1) makes it possible, after the determination of parameters λ_j, to establish the law of variation of angle γ for which equalities (1.5) and (1.6) are valid.

In the case of a fixed gyropendulum suspension point $\omega_0 = 0,\ v = 0$. Hence $\omega_1 = 0,\ \omega_2 = 0,\ \omega_3 = -aF / H$ and Eqs. (2.2) assume the form of equations obtained in /4/ for a gyropendulum mounted on a fixed base on condition that the introduced there quantity $h = \omega_z' - a(P_z + F_z) / H$ is zero for the chosen law of motion of the coordinate system $Oxyz$ with respect to angle γ.

3. Let us show that there is an analogy between the gyropendulum precession equations and the equations of the basic problem of inertial navigation /5,8,9/.

Let $\xi^s\eta^s\zeta^s$ be a nonrotating coordinate system, and XYZ a system of coordinates attached to the stabilized platform. Orientation of the coordinate system XYZ relative to the trihedron $\xi^s\eta^s\zeta^s$ is defined by the angles $\psi, -\varphi, -\varkappa$ whose meaning is explained in /5/. Conditionally assuming the coordinate axes x, y, z as nonrotating, we superpose these on the axes ζ^s, ξ^s, η^s respectively. It follows from Fig.1 and the scheme of turns of the coordinate trihedron XYZ relative to $\xi^s\eta^s\zeta^s$ presented in /5/ that when the equalities

$$\alpha = \varkappa,\ \beta = \varphi,\ \gamma = -\psi \tag{3.1}$$

are satisfied, the coordinate axes $x^\circ, y^\circ, z^\circ$ coincide with axes Z, X, Y, respectively, and the projections $\omega_x, \omega_y, \omega_z$ of the absolute angular velocity of rotation of the trihedron XYZ about its own axis are defined by

$$\omega_x = -\omega_2,\ \omega_y = -\omega_3,\ \omega_z = -\omega_1 \tag{3.2}$$

From Eqs. (1.3) and equalities (3.1) and (3.2) we obtain equations

$$-\varphi^\cdot \cos\varkappa - \psi^\cdot \cos\varphi \sin\varkappa = \omega_x,\quad -\varphi^\cdot \sin\varkappa + \psi^\cdot \cos\varphi^\cdot \cos\varkappa = \omega_y \tag{3.3}$$
$$-\varkappa^\cdot + \psi^\cdot \sin\varphi = \omega_z$$

which are the same as the equations of the basic problem of inertial navigation in terms of Euler–Krylov angles /5,8,9/.

Liapunov stability of solutions of Eqs. (3.3) for any continuous functions $\omega_x, \omega_y, \omega_z$ was proved in /10–14/. Equations (1.3) together with formulas (3.1) and (3.2) are equivalent to Eqs. (3.3). Equations (1.9) of the gyropendulum are obtained from Eqs. (1.3) as the result of their solution for the derivatives $\alpha^\cdot, \beta^\cdot, \gamma^\cdot$ and the substitution of expressions (1.8), whose continuity is obvious, for ω_i. Consequently, the solutions of Eqs. (1.9) of the gyropendulum in terms of Euler–Krylov angles α and β are also Liapunov stable.

The Rodrigues–Hamilton parameters $l_j\ (j = 0, 1, 2, 3)$ that define the position of the coordinate system XYZ relative to $\xi^s\eta^s\zeta^s$ are related to parameters λ_j by formulas

$$l_0 = \lambda_0,\ l_1 = -\lambda_2,\ l_2 = -\lambda_3,\ l_3 = -\lambda_1 \tag{3.4}$$

From equalities (3.2) and (3.4) and Eqs. (2.2) we obtain the equations

$$2l_0^\cdot = -(\omega_x l_1 + \omega_y l_2 + \omega_z l_3),\quad 2l_1^\cdot = \omega_x l_0 + \omega_z l_2 - \omega_y l_3 \tag{3.5}$$
$$2l_2^\cdot = \omega_y l_0 - \omega_z l_1 + \omega_x l_3,\quad 2l_3^\cdot = \omega_z l_0 + \omega_y l_1 - \omega_x l_2$$

that coincide with the equations of the basic problem of inertial navigation in terms of the Rodrigues–Hamilton parameters /5/.

In studies of inertial navigation /12,14–17/ another set of Rodrigues–Hamilton parameters m_j is used, as a rule, since it leads to simpler relations between parameters m_j and angles $\psi, \varphi, \varkappa$. The relation between parameters l_j and m_j was established in /5/, where it was shown that the equations of the basic problem of inertial navigation in terms of parameters m_j are of the same form as Eqs. (3.5) (with the substitution of m_j for l_j).

The Liapunov stability of solutions of Eqs. (3.5) for any continuous functions $\omega_x, \omega_y, \omega_z$ was proved in /12/. Equations /2.2/ are equivalent to Eqs. (3.2), (3.4) and (3.5) when ω_i are defined by the continuous functions (1.8), and are equations of the gyropendulum precession in terms of the Rodrigues–Hamilton parameters. It is therefore, possible to conclude that solutions of Eqs. (2.2) and (1.8) of the gyropendulum in terms of the Rodrigues–Hamilton parameters are Liapunov stable.

The established analogy between precession equations of the gyropendulum and the equations of the basic problem of inertial navigation enable us to state that the solutions of Eqs. (1.9) of the gyropendulum in terms of Euler–Krylov angles, as well as of Eqs. (2.2) and (1.8) in terms of Rodrigues–Hamilton parameters are Liapunov stable. One must, however, bear in mind that the values

$$\alpha^* = 0,\ \beta^* = 0,\ \gamma^* = \gamma^*(t)\ (\gamma^{*\cdot} = \omega_3)$$

of variables α, β, γ and the respective values

$$\lambda_0^* = \cos(\gamma^*/2),\ \ \lambda_1^* = \lambda_2^* = 0,\ \ \lambda_3^* = \sin(\gamma^*/2)$$

of variables λ_j for which the principal axis of the gyropendulum coincides with the vertical, are not particular solution of Eqs. (1.9) in terms of Euler–Krylov angles and of Eqs. (2.2) and (1.8) in terms of the Rodrigues–Hamilton parameters. Because of this, the conclusion on the Liapunov stability of equations solutions of the gyropendulum precession does not imply motion stability of the gyropendulum principal axis with respect to the vertical. This problem requires separate consideration. The problem of unambiguous determination of angles α, β, γ in terms of the Rodrigues–Hamilton parameters λ_j must, also, be considered. For this, the method described in /5/ should be used, taking into account that α, β, γ may assume any values in the intervals $(-\pi/2, \pi/2)$, $(-\pi/2, \pi/2)$, and $(0, 2\pi)$, respectively.

Structure of the general solution of system (2.2) is given in /5,7,16/. The solutions of Eqs. (2.2) are also known for particular cases of specification of vector ω, for instance, for vector ω whose direction in the coordinate system $Ox^\circ y^\circ z^\circ$ is constant, and for vector ω effecting a conical motion. Hence for the establishment of the structure of the general solution of system (1.9), as well as for the cases of specifications of the angular velocity vector ω it is necessary to use formulas (2.1).

4. Let us derive the linear differential equations defining the precession motion of a gyropendulum relative to the geographical trihedron /2,3/ in finite angles.

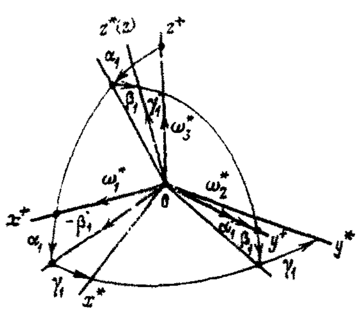

Fig.2

Let $Ox^+y^+z^+$ be a geographic system of coordinates whose axis Oz^+ coincides with Oz°-axis of the Darboux trihedron $Ox^\circ y^\circ z^\circ$ and is directed along the terrestrial sphere radius, and the Ox^+- and Oy^+-axes point to the East and North, respectively. We attach to the gyropendulum rotor axis z the coordinate system $Ox^*y^*z^*$ whose Oz^*-axis is directed along the rotor axis. Position of the coordinate system $Ox^*y^*z^*$ relative to the geographical trihedron $Ox^+y^+z^+$ is defined by angles $\alpha_1, \beta_1, \gamma_1$ (Fig.2). The angles α_1 and β_1 determine the position of the gyropendulum rotor axis z relative to the trihedron $Ox^+y^+z^+$.

The angular velocity ω^* of rotation of the coordinate system $Ox^*y^*z^*$ relative to the trihedron $Ox^+y^+z^+$ is in conformity with the vector equation (1.7) of the form

$$\omega^* = a(\mathbf{F} - m\mathbf{w}) / H - \omega^+ \tag{4.1}$$

where ω^+ is the absolute angular velocity of the geographical trihedron.

For the projections ω_i^* of vector ω^* on the axes of the coordinate system $Ox^+y^+z^+$ we obtain the following formulas:

$$\omega_1^* = -amw_1 / H - u_1,\ \ \omega_2^* = -amw_2 / H - u_2,\ \ \ \omega_3^* = -a(F + mw_3) / H - u_3 \tag{4.2}$$

where w_i and u_i are projections of vectors $\mathbf{w}$ and ω^+ on the axes of the coordinate system $Ox^+y^+z^+$, which are defined by formulas appearing in /2,3/.

Projections ω_i^* may also be represented in the form (see Fig.2)

$$\omega_1^* = -\beta_1^\cdot \cos\alpha_1 + \gamma_1^\cdot \cos\beta_1 \sin\alpha_1,\ \ \omega_2^* = \alpha_1^\cdot + \gamma_1^\cdot \sin\beta_1,\ \ \ \omega_3^* = \beta_1^\cdot \sin\alpha_1 + \gamma_1^\cdot \cos\beta_1 \cos\alpha_1 \tag{4.3}$$

Using expressions (4.2) we solve Eqs. (4.3) for the derivatives $\alpha_1^\cdot, \beta_1^\cdot, \gamma_1^\cdot$ and obtain the formulas given in /2,3/ for equations of precession motion of the gyropendulum rotor axis z relative to the geographical trihedron

$$H(\alpha_1^\cdot \cos\beta_1 - u_1 \sin\alpha_1 \sin\beta_1 + u_2 \cos\beta_1 - u_3 \cos\alpha_1 \sin\beta_1) = \tag{4.4}$$
$$a(F + mw_3)\cos\alpha_1 \sin\beta_1 + amw_1 \sin\alpha_1 \sin\beta_1 - amw_2 \cos\beta_1$$
$$H(\beta_1^\cdot - u_1 \cos\alpha_1 + u_3 \sin\alpha_1) = -a(F + mw_3)\sin\alpha_1 + amw_1 \cos\alpha_1$$

and the equation

$$\gamma_1^{\cdot} = -\frac{1}{\cos\beta_1}\left\{\left(\frac{a}{H}mw_1 + u_1\right)\sin\alpha_1 + \left[\frac{a}{H}(F + mw_3) + u_3\right]\cos\alpha_1\right\}$$

which, after the determination of the unknown functions $\alpha_1(t)$, $\beta_1(t)$ from system (4.4), make possible the establishment of the law of motion of the coordinate system $Ox^*y^*z^*$ in terms of angle γ_1.

We attach the solid body D^* to the coordinate system $Ox^*y^*z^*$ and introduce the finite turn vector θ^* which determines the position of body D^* relative to the trihedron $Ox^+y^+z^+$. We denote by $\nu_j\ (j = 0, 1, 2, 3)$ the Rodrigues—Hamilton parameters that correspond to the finite turn vector θ^*. The kinematic equations of the spherical motion of body D^* relative to the trihedron $Ox^+y^+z^+$ in terms of Rodrigues—Hamilton parameters ν_j are

$$2\nu_0^{\cdot} = -(\omega_1^*\nu_1 + \omega_2^*\nu_2 + \omega_3^*\nu_3),\quad 2\nu_1^{\cdot} = \omega_1^*\nu_0 + \omega_2^*\nu_3 - \omega_3^*\nu_2 \tag{4.5}$$
$$2\nu_2^{\cdot} = \omega_2^*\nu_0 + \omega_3^*\nu_1 - \omega_1^*\nu_3,\quad 2\nu_3^{\cdot} = \omega_3^*\nu_0 + \omega_1^*\nu_2 - \omega_2^*\nu_1$$

when coefficients ω_i^* conform to formulas (4.2), define in the geographical coordinate system the gyropendulum behavior for finite angles of the rotor axis deflection from the vertical.

Thus for the determination of angles α_1 and β_1 which define the position of the gyropendulum rotor axis z relative to the geographical trihedron for arbitrary motion of its base over the Earth surface it is possible to solve, instead of two nonlinear differential equations (4.4), four linear differential equations (4.5) passing from parameters ν_j to angles α_1 and β_1 in conformity with formulas

$$\operatorname{tg}\alpha_1 = 2(\nu_0\nu_2 + \nu_1\nu_3)\,/\,(\nu_0^2 + \nu_3^2 - \nu_1^2 - \nu_2^2),\quad \sin\beta_1 = 2(-\nu_0\nu_1 + \nu_2\nu_3)$$

derived similarly to (2.1).

In conclusion, we would point out that the inference about Liapunov stability of solutions of Eqs. (1.9), and (2.2) and (1.8) of precession motion of the gyropendulum rotor axis relative to the accompanying Darboux trihedron, arrived at in Sect.3, as well as those about the particular cases of integrability of these equations in quadratures, can be extended to Eqs. (4.4), and (4.5) and (4.2) of precession motion of the gyropendulum rotor axis relative to the geographical trihedron.

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